expansion as $\Sigma(p_{i_1} p_{i_2}+1 \ldots \ldots p_{i_n}+n-1)$, where $i_1, i_2 \ldots\ldots i_n$ form a permutation of the numbers 1, 2, $\ldots\ldots n$. Thus we have the formula

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=[p_1 p_2 \ldots\ldots\ p_n]=\Sigma(p_{i_1}p_{i_2}+1\ldots\ldots\ p_{i_n}+n-1).$$

(2). *Application of the Formula.*

To apply it we notice that the matrices

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \ldots\ldots \alpha_1^{2n-1} \\ 1 & \alpha_2 & \alpha_2^2 \ldots\ldots \alpha_2^{2n-1} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ 1 & \alpha_n & \alpha_n^2 \ldots\ldots \alpha_n^{2n-1} \end{vmatrix} \text{ and } \begin{vmatrix} a_n & a_{n-1} & a_{n-2}\ldots\ldots a_0 & 0 & 0\ldots\ldots 0 \\ 0 & a_n & a_{n-1}\ldots\ldots a_1 & a_0 & 0\ldots\ldots 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0\ldots 0 & a_n\ldots\ldots & \ldots\ldots & \ldots\ldots a_1 a_0 \end{vmatrix}$$

correspond, and from the theorem that "the corresponding determinants of corresponding matrices are proportional,"* that

$$D=\rho \begin{vmatrix} a_0 & 0 & 0\ldots\ldots 0 \\ a_1 & a_0 & 0\ldots\ldots 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n-1} & \ldots\ldots & a_1 a_0 \end{vmatrix} = \rho a_0^n.$$

When, as often happens, any of the numbers p_r+i_r, p_s+i_s, in a parenthesis, are equal, such symbolic parenthesis reduces to zero, because the determinant thereby signified has at least two columns equal. The values of the non-vanishing symbols are then read off from the second matrix by the help of the theorem concerning corresponding matrices, the factor ρ being added and the correct sign factor of the corresponding determinant. The factor ρ will then divide out and the result gives

$$a_0^n \Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$$

which equals the algebraic sum of a number of determinants in the a's of the nth order.

(3). *Proof of the Formula.*

I have proved this theorem as follows: We will begin by considering some simple examples.

a. It is required to find $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}$ for a quadratic equation $a_0x^2+a_1x+a_2=0$. The matrices are

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 \end{vmatrix} \text{ and } \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \end{vmatrix}.$$

$$D=\begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix} = \rho \begin{vmatrix} a_0 & 0 \\ a_1 & a_0 \end{vmatrix} = \rho a_0^2.$$

*Gordan, Determinants, p. 95.

We indicate $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}=(p_1p_2)$. We perform the operation indicated on the left hand side in detail. We have

$$(\alpha_2-\alpha_1)(\alpha_1^{p_1}\alpha_2^{p_2}+\alpha_1^{p_2}\alpha_2^{p_1})=\alpha_1^{p}\alpha_2^{p_2+1}+\alpha_1^{p_2}\alpha_2^{p_1+1}-\alpha_1^{p_1+1}\alpha_2^{p_2}-\alpha_1^{p_2+1}\alpha_2^{p_1}$$

$$=\begin{vmatrix}\alpha_1^{p_1}\alpha_1^{p_2+1}\\ \alpha_2^{p_1}\alpha_2^{p_2+1}\end{vmatrix}+\begin{vmatrix}\alpha_1^{p_2}\alpha_1^{p_1+1}\\ \alpha_2^{p_2}\alpha_2^{p_1+1}\end{vmatrix}$$

by taking the first and fourth, and the second and third terms together. If we had multiplied

$$\begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix} \text{ by } \Sigma\alpha_1^{p_1}\alpha_2^{p_2} \text{ as } \begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix}\alpha_1^{p_1}\alpha_2^{p_2}+\begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix}\alpha_1^{p_2}\alpha_2^{p_1},$$

we should have obtained

$$\begin{vmatrix}\alpha_1^{p_1}\alpha_1^{p_2+1}\\ \alpha_2^{p_1}\alpha_2^{p_2+1}\end{vmatrix}+\begin{vmatrix}\alpha_1^{p_2}\alpha^{p_2+1}\\ \alpha_2^{p_1}\alpha^{p_1+1}\end{vmatrix}.$$

If we compare with the previous result, we see that we could convert one into the other by exchanging in each the two secondary diagonal terms which is permitted. We choose the result first obtained and symbolize it by $(p_1p_2+1)+(p_2p_1+1)$, where in (p_1p_2+1) p_1 is the exponent of the first, and p_2+1 that of the second column of the determinant for which it stands. We write completely

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}=(p_1p_2)=(p_1p_2+1)+(p_2p_1+1).$$

b. We will apply this formula to the calculation of the symmetric functions required by the resultant of two quadratic forms,

$$fx=a_0x^2+a_1x+a_2=a_0(x-\alpha_1)(x-\alpha_2).$$
$$\phi x=b_0x^2+b_1x+b_2=b_0(x-\beta_1)(x-\beta_2).$$

The resultant of these forms is

$$a_0^2(b_0\alpha_1^2+b_1\alpha_1+b_2)(b_0\alpha_2^2+b_1\alpha_2+b_2)=$$
$$a_0^2(b_0^2\Sigma\alpha_1^2\alpha_2^2+b_0b_1\Sigma\alpha_1^2\alpha_2+b_0b_2\Sigma\alpha_1^2+b_1^2\Sigma\alpha_1\alpha_2+b_1b_2\Sigma\alpha_1+b_2^2).$$

c. We comprehend the results together in the following table. The sign of the determinant which corresponds to $(\alpha_1^{\lambda}\beta_1^{\mu})$ is by the theorem of corresponding matrices,

$$(-1)^{(1+2)+(\lambda+1+\mu+1)}=(-1)^{\lambda+\mu+1}.$$

We also observe that $(\lambda\lambda)=0$, and do not write 0.

Function	$[p_1 p_2]$	$\begin{matrix}(p_1 & p_2+1)\\(p_2 & p_1+1)\end{matrix}$	Determinants	Sign	Result
$2a_0^2\Sigma\alpha_1^2\alpha_2^2$	[2 2]	2(2 3)	$2\begin{vmatrix} a_2 & a_1 \\ 0 & a_2 \end{vmatrix}$	$(-1)^6=1$	$2a_2^2$
$a_0^2\Sigma\alpha_1^2\alpha_2$	[2 1]	$\begin{matrix}(2\ 2)\\(1\ 3)\end{matrix}$	$\begin{vmatrix} a_2 & a_0 \\ 0 & a_1 \end{vmatrix}$	$(-1)^5=-1$	$-a_1a_2$
$a_0^2\Sigma\alpha_1^2$	[2 0]	$\begin{matrix}(2\ 1)\\(0\ 3)\end{matrix}$	$-\begin{vmatrix} a_2 & 0 \\ 0 & a_0 \end{vmatrix}$ $\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}$	$-(-1)^4=-1$ $(-1)^4=1$	$-2a_0a_2+a_1^2$
$2a_0^2\Sigma\alpha_1\alpha_2$	[1 1]	2(1 2)	$2\begin{vmatrix} a_2 & 0 \\ 0 & a_0 \end{vmatrix}$	$(-1)^4=1$	$2a_2a_0$
$a_0^2\Sigma\alpha_1$	[1 0]	$\begin{matrix}(1\ 1)\\(0\ 2)\end{matrix}$	$\begin{vmatrix} a_1 & 0 \\ a_2 & a_0 \end{vmatrix}$	$(-1)^3=-1$	$-a_0a_1$

d. Substituting these values we have for the final result the function of the coefficients of the two quadratics

$$b_0^2a_2^2-b_0b_1a_1a_2+b_0b_2(a_1^2-2a_0a_2)+b_1^2a_0a_2-b_1b_2a_0a_1+b_2^2a_0^2.$$

e. We will now take another step and consider $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. We wish to show that we obtain the same result when we apply the exponents in all possible permutations to the columns of D that we obtain when we apply them to the rows, and obtain $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. To prove this we need only show that to every term of the second formation corresponds exactly the same term in the first formation. First, it is clear that the number of terms in each formation is $(3!)^2=36$. Next, take a term like $\alpha_1^{p_3+1}\alpha_2^{p_1+2}\alpha_3^{p_2}$ of the second formation. It must have come from

$$\begin{vmatrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \\ \alpha_3^0 & \alpha_3 & \alpha_3^2 \end{vmatrix} \begin{matrix} p_3 \\ p_1=d' \\ p_2 \end{matrix}$$

which signifies that D has been multiplied by $\alpha_1^{p_3}\alpha_2^{p_1}\alpha_3^{p_2}$ of $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$, the first row by $\alpha_1^{p_3}$, the second by $\alpha_2^{p_1}$, and the third by $\alpha_3^{p_2}$, and here as the exponents come to be applied to the rows as the result of the multiplication, they are legitimately applied. Again write D changing columns into rows, and obtain

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}$$

Next inquire by what substitution $\alpha_1^{p_3+1}\alpha_2^{p_1+2}\alpha_3^{p_2}$ came from the previous determinant d'. We see that it corresponds to 1st line 2d column, 2d line 3d column, 3d line 1st column, or to the substitution $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Now in order that we may arbitrarily affix the exponents to the rows of D as last written, which is the same as affixing them to the columns as before written, and obtain this term, at least numerically, we must have p_3 with the second line, p_1 with the third, and p_2 with the first. Thus we must have

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} \begin{matrix} p_2 \\ p_3 = d_1 \\ p_1 \end{matrix}$$

which signifies that the exponents p are to be arbitrarily affixed to those of the corresponding line.

What substitution now gives the term in this determinant? We see that it corresponds to 2d line 1st column, 3d line, 2d column, 1st line 3d column, or to $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$; and this last substitution is the reciprocal of the preceding $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and has the same sign with it. We see farther that to every term of d' corresponds one and only one determinant like d_1, and in d_1 there is one and only one term equal to the given term of d'. To the six terms of d' correspond the six determinants of $d_1, d_2, \ldots\ldots d_6$; to the six terms of d'' correspond the same six determinants $d_1, \ldots\ldots d_6$, and so on; to the 36 terms of the six determinants $d', d'', \ldots\ldots d^{\text{vi}}$, correspond the six determints $d_1, \ldots\ldots d_6$, and conversely to the 36 terms of the six determinants $d_1, \ldots\ldots d_6$, correspond the six determinants $d', d'', \ldots\ldots d^{\text{vi}}$. Thus the 36 terms of the one set are equal to the 36 terms of the other set, and the results of the two operations are identical.

f. We may express the result of this example more briefly if we write:

$$\begin{vmatrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \\ \alpha_3^0 & \alpha_3 & \alpha_3^2 \end{vmatrix} \begin{matrix} p_{i_1} \\ p_{i_2} \\ p_{i_3} \end{matrix}$$

and take any one of the 36 terms obtained by applying the 3! substitutions $\begin{pmatrix} 1 & 2 & 3 \\ j_1 & j_2 & j_3 \end{pmatrix}$ where the j's are the numbers of the columns, to the six determinants that arise from the 3! permutations of the i's which form a permutation of the numbers 1, 2, 3. We obtain a term of the form

$$\alpha_1^{p_{i_1}+j_1-1}\alpha_2^{p_{i_2}+j_2-1}\alpha_3^{p_{i_3}+j_3-1}.$$

[To be continued.]

ENCYKLOPAEDIE DER MATHEMATISCHEN WISSENSCHAFTEN.

By DR. GEORGE BRUCE HALSTED.

Mit Unterstuetzung der Akademicen der Wissenschaften zu Muenchen und Wien und der Gesellschaft der Wissenschaften zu Goettingen, herausgegeben von *H. Burkhardt* und *W. F. Meyer*. Band I. Heft 1. Leipzig, Teubner. 1898. Pages 1—112.

This is an undertaking of extraordinary importance and promise. Its aim is to give a conservative presentation of the assured results of the mathematical sciences in their present form, while, by careful and copious references to the literature, giving full indications regarding the historic development of mathematical methods since the beginning of the nineteenth century. The work begins with 27 pages on the foundations of arithmetic by Hermann Schubert of Hamburg. Schubert's reputation was made by his remarkable book on enumerative geometry. He has since applied modern ideas in an elementary arithmetic, and is known in America as a contributor to the *Monist*. Unfortunately, Schubert has made in public some strange slips. In an article "On the nature of mathematical knowledge," in the *Monist*, Vol. 6, page 295, he says: "Let me recall the controversy which has been waged in this century regarding the eleventh axiom of Euclid, that only one line can be drawn through a point parallel to another straight line. The discussion merely touched the question whether the axiom was capable of demonstration solely by means of the other propositions or whether it was not a special property, apprehensible only by sense-experience, of that space of three dimensions in which the organic world has been produced and which therefore is of all others alone within the reach of our powers of representation. The truth of the last supposition affects in no respect the correctness of the axiom but simply assigns to it, in an epistemological regard, a different status from what it would have if it were demonstrable, as was one time thought, without the aid of the senses, and solely by the other propositions of mathematics."

If Schubert had written this seventy-five years ago it might have been pardonable. Just at the beginning of this century Gauss was trying to prove this Euclidean parallel-postulate. Even up to 1824 he was in Schubert's state of mind, for he then writes Taurinus: "Ich habe daher wohl zuweilen *in Scherz* den Wunsch geaeussert, dass die Euclidische Geometrie nicht die Wahre waere." But the joke had even then gone out of the matter if Gauss had but known it, for in 1823 Bolyai Janos had written to his father, "from nothing I have created a wholly new world." Of the geometry of this world as given also by Lobachevski, Clifford wrote: "It is quite simple, merely Euclid without the vicious assumption." But this assumption is only vicious if supposed to be "apprehensible by sense-experience" or "demonstrable by the aid of the senses." That "the organic world has been produced" in Euclidean space can never be demonstrated

in any way whatsoever. On the other hand, the mechanics of actual bodies might be shown by merely approximate methods to be non-Euclidean.

Therefore Schubert's contribution on the foundations of arithmetic may fairly be read critically. He begins with counting, and defines number as the result of counting. This is in accord with the theory that their laws alone define mathematical operations, and the operations define the various kinds of number as their symbolic outcome. There is no word of the primitive number-idea, which is essentially prior to counting and necessary to explain the cause and aim of counting. This primitive number-idea is a creation of the human mind, for it only pertains to certain other creations of the human mind which I call artificial individuals. The world we consciously perceive is a mental phenomenon. Yet certain separable or distinct things or primitive individuals we cannot well help believing to subsist somehow 'in nature' as well as in conscious perception. Now by taking together certain of these permanently distinct things or natural individuals the human mind makes an artificial individual, a conceptual unity.

Number is primarily a quality of such an artificial individual. The operation of counting was made to apply to such an individual to identify it with one of a standard set of such artificial individuals, and so to get the exact shade of its numeric quality. These standard individuals were primarily sets of fingers. Then came the written standard set, *e. g.* III, or 1+1+1; and finally the written symbol 3. Such symbols serve to represent and convey the numeric quality. The word number is applied indiscriminately to the quality or idea and to its symbol.

Schubert tells us that in antiquity the Romans represented the numbers from one to nine by rows of strokes, as 4 is still represented on our watches; while the Aztecs used to put together single circles for the numbers from one to nineteen. I have seen Japanese use columns of circles in the same way. Thus also our striking clocks convey a numeric quality by a group possessing it. But the number pertaining to a group or artificial individual is far from being the simple notion it seems. If numbers are used to express exactly this definite attribute of finite systems they are called cardinal numbers.

Schubert's first sentence is: Dinge *zaehlen* heisst, sie als gleichartig ansehen, zusammen auffassen, und ihnen einzeln andere Dinge zuordnen, die man auch als gleichartig ansieht. This may be rendered: "*To count* things means, to consider them as alike, to take them together, and to associate them singly to other things which one also considers as alike." I would prefer to say: "To count distinct things means to make of them an artificial individual or group and then to identify its elements with those of a familiar group."

When the mind of man made these artificial individuals, they were found to possess a sort of property or quality which was independent of the distinctive marks of the natural individuals composing them, also independent of the order or sub-association of these natural individuals. Whether the artificial individual were made of a church, a noise, and a pain, or made of three peas, or composed of two eyes and a nose, it had one certain quality, it was a triplet.

I see no necessity for Schubert to consider the church as like the noise and the pain. Again, the individuals of the familiar group used in the count need not be alike. Even the individuals used by a clock in counting differ ordinarily, and when we follow the count of the clock we use words all different. The primitive written number is such a picture of a group of individuals as represents their individual existence and nothing more, *e. g.* III; so however different they may be, this number is independent of the order in which they are associated with its elements.

Schubert wastes three sentences on the so-called concrete number, *benannte* Zahl. Three quails is not a number, but is a particular bevy.

His §2 *Addition*, he begins thus: "If one has two groups of units such that not only all units of each group are alike, but that also each unit of the one group is like each unit of the other group, etc." All this likeness and alikeness seems unnecessary. Any two groups may be thought into one group. Any two primitive numbers may be added.

In section 5, Peacock's Principle of Permanence is given it Hankel's general form: The combination of two numbers by any defined operation is a number, such that the combination may be handled as if it gave one of the previously defined numbers. New kinds of numbers, like all numbers, are defined by the operations from which they result. Thus are introduced zero and negative numbers, and later, the fraction. After this all is easy to the end of Schubert's contribution. It only remains to point out, as of especial importance, that from beginning to end not the slightest mention is made of measurement. Not a word is wasted on people who do not clearly see that number is long prior to measurement.

The second section of the Encyklopaedie is "Kombinatorik" by E. Netto. This is a part of mathematics which never fulfilled the hopes of the school which was lost in it during the early part of this century. Of the most comprehensive monographs the last two are in 1826 and 1837. For us it has gone over into determinants, and more than half of Netto's article is devoted to determinants. This article is particularly valuable from a bibliographic and historic point of view.

The third section is "Irrationalzahlen und Konvergenz unendlicher Prozesse," by A. Pringsheim. It begins on page 47, and goes past the end of the Heft. This is a modern subject, of intense living interest. How entirely modern it is might not be suspected by readers of such sentences in Cajori's excellent history of mathematics as those on page 70; "The first incommensurable ratio known seems to have been that of the side of a square to its diagonal, as $1:\sqrt{2}$. Theodorus of Cyrene added to this the fact that the sides of squares represented in length by $\sqrt{3}$, $\sqrt{5}$, etc., up to $\sqrt{17}$, and Theaetetus, that the sides of any square, represented by a surd, are incommensurable with the linear unit."

Now in fact Theodorus and Theaetetus made no representation whatever of the length of these sides, simply saying, *e. g.* that the side of the square whose area is 3 is incommensurable with the side of the square whose area is one. For

Euclid there was no such ratio as $1:\sqrt{2}$; for 1 is a number and so if it could have had a ratio to $\sqrt{2}$ this would have been a number. But Euclid, Book X, Proposition 7 is; "Incommensurable magnitudes are not to one another in the ratio of one number to another number." The Hindus were the first to recognize the existence of irrational numbers. Even through the middle ages and the renaissance they were absurd fictions, "numeri surdi," a designation attributed to Leonardo of Pisa. The first writer to genuinely treat them was Stifel (1544), and even he had not completely freed himself from the older terminology, since he says: "sic irrationalis numerus non est verus atque lateat sub quadam infinitatis nebula." In reference to the next step, the conceiving of ratio as number, Pringsheim says, page 51, "Hatte schon *Descartes* beliebige *Streckenverhaeltnisse* mit *einfachen Buchstaben* bezeichnet, und damit *wie mit Zahlen* gerechnet, etc." Bnt here I think the careful German has slipped.

In regard to just this point a common error is still widespread, which we see in the following, read before Sections A and B of the American Association for the Advancement of Science, 1891:

"The doctrine of Descartes was, that the algebraic symbol did not represent a concrete magnitude, but a mere number or ratio, expressing the relation of the magnitude to some unit. Hence that the product of two quantities is the product of ratios, . . . ; that the powers of a quantity are ratios like the quantity itself," etc.

That every statement here quoted is a mistake will be instantly seen from the following, taken from pages numbered 297—9 of the *original edition* of Descartes' Geometrie, 1637, a copy of which (perhaps unique on this continent) I have had the good fortune to possess since my student days (1876).

"Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, que sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision: Ainsi n' at' on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres, ou en oster; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer que quatriesme. qui soit a l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité est a l'autre, ce qui est le mesme que la Diuision; au enfin trouuer vne, ou deux, ou plusieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui est le mesme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligible. * * *

"Mais souuent on n'a pas besoin de tracer ainsi ceè lignes sur le papier, & il suffist de les designer par quelques lettres, chascune par vne seule. Comme pour adiouster la ligne BD a GH, ie nomme l'vne a & l'autre b, & escris $a+b$; Et $a-b$, pour soustraire b d' a; Et ab, pour les multiplier l'vne par l' autre; Et $\frac{a}{b}$, pour diuiser a par b; Et aa, ou a^2, pour multiplier a par soy mesme; Et a^3, pour le multiplier encore vne fois par a, & ainsi a l' infini; Et $\sqrt{a^2+b^2}$, pour tirer la racine quarrée d' a^2+b^2; Et $\sqrt{C.a^3-b^3+abb}$, pour tirer la racine cubique d' a^3-b^3+abb, & ainsi des autres.

"Qu il est a remarquer que par a^2 ou b^3 ou semblables, ie ne concoy ordinairement que des lignes toutes simples, encore que pour me seruir des noms vsités en l'Algebre, ie les nomme des quarrés, ou des cubes, &c."

Thus what Descartes really did was to make a geometric algebra, in which, however, the product of two *sects* (Strecken) was not a rectangle but a sect; the product of three sects not a cuboid but a sect. Here Descartes represents by the single letters *a*, *b*, sects, Strecken, not *Streckenverhaeltnisse*. Descartes does not here pass beyond Euclid's representation of the ratio of two magnitudes by two other magnitudes, does not reach the conception of the systematic representation of the ratio of two magnitudes by one magnitude, that one magnitude to be always interpreted as a number. This radical innovation, the creation of this epoch-marking paradox, is due to Newton. Newton takes this vast step explicitly and consciously. The lectures which he delivered as Lucasian professor at Cambridge were published under the title "Arithmetica Universalis." At the beginning of his Arithmetica Universalis he says, page 2, "Per Numerum non tam multitudinem unitatum quam abstractam quantitatis cujusvis ad aliam ejusdem generis quantitatem quae pro unitate habetur rationem intelligimus." [In quoting this, Pringsheim, page 51, misses the first word. He omits the *Per.*] As Wolf puts it, (1710) "Number is that which is to unity as a piece of a straight line [a sect] is to a certain other sect." Thus the length of any sect is a real number, and the length of any possible sect incommensurable with the unit sect is an irrational number. Says Hayward in his Vector Algebra (1892), page 5, "Number is essentially *discrete* or *discontinuous*, proceeding from one value to the next by a finite increment or jump, and so cannot, except in the way of a limit, represent, relatively to a given unit, a continuous magnitude for which the passage from one value to another may always be conceived as a *growth* through every intermediate value." But the moment we accept Newton's definition of number it takes on whatever continuity is possessed by the sect. However, from this alone does not follow that for every irrational there is a sect whose length would give that irrational. G. Cantor was the first to bring out sharply that this is neither self-evident nor demonstrable, but involves an essential pure geometric assumption. To free the foundations of general arithmetic from such *geometric* assumption, G. Cantor and Dedekind each developed his pure arithmetic theory of the irrational. Professor Fine in his "Number-System of Algebra" seems to miss this point completely. He gives, page 42, what purports to be a demonstration that "Corresponding to every real number is a point on the line, the distance of which from the null-point is represented by the number," without any mention of the geometric assumption necessary, and then proceeds, page 43, to borrow the continuity of his number system from the näively supposed continuity of the line, the very thing for the avoidance of which G. Cantor and Dedekind made their systems. Says Dedekind, "Um so schoener erscheint es mir, dass der Mensch ohne jede Vorstellung von messbaren Groessen, und zwar durch ein endliches System einfacher Denkschritte sich sur Schoepfung des reinen, stetigen Zahlenreiches aufschwingen kann; und erst mit diesem Huelfsmittel wird es ihm nach meiner Ansicht moeglich, die Vorstellung von stetigen Raume zu einer deutlichen auszubilden."

Austin, Texas.

718121

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

103. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Find the proceeds of a note discounted at a bank for 10 years at 10%. What is the meaning of the result?

I. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

The proceeds would be nothing. The meaning is that the method is erroneous and unjust. It was invented to evade the usury laws. I have thought that the court which first sustained the method could not have been well versed in mathematical principles.

II. Solution by the PROPOSER.

The bank discount upon any sum exceeds the true discount for the same time and rate, by the interest upon the true discount for the given time. Bank discount then is of the nature of compound interest, for a banker in discounting a note not only deducts the interest which is to accrue upon it from that time until the date of maturity, but also charges a certain per cent. for his services, which per cent. is the interest upon the interest which is to accrue. Now the true discount of any sum for 10 years at 10 per cent. is just half that sum, and as the amount will double itself at simple interest for 10 years at 10 per cent., the discount which the banker deducts for that time, plus the interest which he charges, is exactly equal to the face of the note, neglecting the three days of grace. The note in this light is worthless in spite of the fact that its present worth is half of its face. The case is of course a very improbable one but it is simply an illustration of the rapidity with which compound interest will accumulate, although the interest here is not compound in strict sense but accumulates slower.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; ELMER SCHUYLER, High Bridge, N. J.; CHAS. C. CROSS, Libertytown, Md., and ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Ia.

Days of grace are no longer in use. Since bank discount is simple interest charged when loan is made the proceeds would be $0.00.

This means that the bank would get twice the amount loaned, while the borrower would have the use of no money. Such a transaction is hardly possible between business men.

Solutions of problem 102 were received too late for credit in the December number from *JOSIAH H. DRUMMOND, P. S. BERG, ALOIS F. KOVARIK, P. H. PHILBRICK, ELMER SCHUYLER, GUY B. COLLIER, G. BRECKENRIDGE,* and *WALTER H. DRANE.*

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithica, N. Y.

Solve by quadratics, $x^2+y=7$.....(1),
$x+y^2=11$.....(2).

IX. Solution by S. F. NORRIS, Professor of Mathematics and Astronomy, Baltimore City College, Baltimore, Md.

$x^2+y=7$.....(1), $x+y^2=11$.....(2).

Eliminating y, $x^4-14x^2+x+38=0$.....(3).

As in many other cases of this nature, resort must be had to special expedients. *Vide Ray's New Higher Algebra*, sec. 253.

Equation (3) may be put in the form

$$x^4-2x^3+2x^3-4x^2-10x^2+20x-19x+38=0$$

by subtracting and adding $2x^3$, separating $-14x^2$ into $-4x^2$ and $-10x^2$, and placing $20x-19x$ for x.

$$x^3(x-2)+2x^2(x-2)-10x(x-2)-19(x-2)=0.$$

$(x-2)(x^3+2x^2-10x-19)=0$, by factoring.

$\therefore x-2=0$.....(a), and $x^3+2x^2-10x-19=0$.....(b).

Hence $x=2$; substituting this value of x in equation (1) or (2), $y=3$.

I cannot discover any special artifice by which the roots of equation (b) can be found by quadratics. By Sturm's Theorem, the three roots of equation (b) are found to lie between 3 and 4, -3 and -2, -1 and -2. By Horner's Method, they are found to be $x_1=3.1313125$, $x_2=-3.283185$, $x_3=-1.84813652$; the corresponding values of y are $y_1=-2.8051181$, $y_2=-3.779310$, $y_3=3.58442837$.

X. Solutions selected by SYLVESTER ROBINS, North Branch Depot, N. J.

(A). From (1), $y=7-x^2$; from (2), $y=\sqrt{(11-x)}$.

$\therefore 7-x^2=\sqrt{(11-x)}$. By squaring, $49-14x^2+x^4=11-x$.

$\therefore x^4-10x^2=4x^2-x-38$.

Add $2x^3-19x=2x^3-19x$, and $x^4+2x^3-10x^2-19x=2x^3+4x^2-20x-38$.

Factoring $x(x^3+2x^2-10x-19)=2(x^3+2x^2-10x-19)$.

$\therefore x=2$, $y=3$.

[W. L. Harvey in "Our Young Folks," *New York Tribune.*]

(B). (1) $x^2+y=11$. (3) $x^2-9=2-y=d$, by assumption.

(2) $x+y^2=7$. (4) $x-3=4-y^2=sd$, by assumption.

Since $x-3=sd$, $(x-3)/s=d$, and $x^2-9=d$, so $x^2-9=(x-3)/s=(x/s)-(3/s)$.

$x^2-(x/s)=9-(3/s)$.

$x^2-(x/s)+(1/4s^2)=9-(3/s)+(1/4s^2)$.

$x-(1/2s)=3-(1/2s)$.

$\therefore x=3$, and $y=2$. [From *The School Day Visitor.*]

NOTE on No. 78, Algebra. Mr. Boorman misapprehends the "real point" of my comment. I will endeavor to make my meaning plain.

Given $x^2+xy=10$ and $y^2+xy=15$ to find the values of x and y. Adding these two equations and taking the square root, we have $x+y=\pm 5$. Dividing the first equation by this, we have $x=\pm 2$, and in the same manner, $y=\pm 3$.

I have supposed that it is *settled* that an equation of the second degree has two roots and *no more*, especially when the equation can be solved by an equation of that degree; and that when an equation of a higher degree, which has more than two roots, is needlessly used in the solution, still the number of roots of the original equation is not thereby increased, and that all, save two, of the roots of the higher equation are not roots of the original equation.

Now, Mr. Boorman, in his solution, page 43 of last volume, by transposing xy to the other side of the original equations and then multiplying the two equations together, obtains an equation of the *fourth* degree. So I said in my Note (and say now) that his solution seems to me to involve the following course of reasoning: If $x^2=36$.....(1), then $x^4=1296$..... (2), and as equation (2) has four roots, therefore equation (1) has four roots: a manifest error as it seems to me.

I did not "mistake him to mean" that the roots of $x^4=16$, are ± 2 and ∓ 2; but in his solution of No. 78, he says there are eight roots, viz.: $x=\mp 2$ and $y=\mp 3$, and $x=\pm 2$ and $y=\pm 3$. But I said that ± 2 and ∓ 2, *taken by themselves*, are precisely the same, and therefore that his *eight* roots are really only *four*, i. e. two pairs, two values of x and two values of y, precisely as are obtained without transforming the original equations of the second degree into a biquadratic equation.

JOSIAH H. DRUMMOND.

GEOMETRY.

105. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Sind A, B, C, D vier harmonische Punkte und beschreibt man über dem Durchmesser AC einen Kreis, von welchem S ein beliebiger Punkt ist, so wird derjenige Kreisbogen, welcher innerhalb des Winkels BSD liegt, entweder von A oder von C halbirt. [*Reye's Geometrie der Lage*, page 191.]

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Sind A, B, C, D vier harmonische Punkte und S ein beliebigen Punkt auf dem Kreis ASC, so sind SA, SB, SC, und SD vier harmonische Strahlen, denn vier harmonische Punkte werden aus jedem Punkte (S) durch vier harmonische Strahlen projicirt.

Weil aber die Strahle SC auf der Strahle SA senkrecht steht, so halbirt sie den Winkel zwischen den anderen beiden, Strahlen SB und SD, und folglich den Kreis-bogen zwischen denselben, nach dem Satz: "Wenn von vier harmonischen Strahlen zwei getrennte auf einander senkricht stehen so halbiren sie die Winkel zwischen den anderen beiden Strahlen."

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let A, B, C, D be the given points.

Join AS, BS, CE, DS. Then $AB:BC=AD:DC$, or $AB:AD=BC:DC$.

$\therefore \triangle ASB:\triangle BSC=\triangle ASD:\triangle CSD$.

$\therefore AS.BS\sin ASB:BS.CS\sin BSC=AS.DS\sin ASD:CS.DS\sin DSC$.

$$\therefore \frac{\sin ASB}{\sin BSC}=\frac{\sin ASD}{\sin DSC}.$$

But $\sin ASB=\sin(\frac{1}{2}\pi-BSC)=\cos BSC$.

$\sin ASD=\sin(\frac{1}{2}\pi+DSC)=\cos DSC$.

$\therefore \cot BSC=\cot DSC$.

$\therefore \angle BSC=\angle DSC$.

$\therefore C$ bisects arc FCE. Similarly A bisects arc $EASF$.

CALCULUS.

80. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A vessel is anchored in three fathoms of water, and the cable passes over a sheave in the bowsprit, which is six feet above the water. If the cable is hauled in at the rate of one foot a second, how fast is the vessel moving through the water when there is five fathoms of cable out? What is the acceleration of the vessel's velocity? [From *Byerly's Problems in Differential Calculus.*] ANS.—(a) 5-6 feet per second; (b) 12-121 feet per second. Are these results correct?

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio

Let x=the horizontal distance from sheave to anchor, and y=the length of cable out, both in feet.

Then $x=(y^2-576)^{\frac{1}{2}}$. Differentiating and making $y=30$ and $dy=-1$, we find $dx=-\frac{5}{4}$; that is, x *diminishes* at the rate of $\frac{5}{4}$ ft. per second, or the ship moves at the same rate. Differentiating again and making the same substitutions we get $d^2x=-\frac{5}{81}$, that is, the vessel's velocity is increasing at the rate $\frac{5}{81}$ ft. per second.

Accordingly the results given as answers to the problem are not correct. It is evident without a solution that the vessel's velocity is always *greater* than the rate at which the cable is hauled in.

II. Solution by ELMER SCHUYLER, High Bridge, N. J.

If I understand the problem, it is (1) $y^2=x^2+24^2$.

We want to find $dx/dt=v$, and $d^2x/dt^2=\alpha$. (2) $y/t=1$.

$\therefore y=t$; $dy=dt$.

$dy/dt=1$, *i. e.* rate is constant and has no acceleration.

$$(3)\ 2x\frac{dx}{dt}=2y\frac{dy}{dt}.$$

(4) $\frac{dx}{dt}=\frac{y}{x}\cdot\frac{dy}{dt}=\frac{30}{\sqrt{30^2-24^2}}\times 1=\frac{5}{3}$ ft. 1st answer.

(5) $\frac{d^2x}{dt^2}=\frac{x(dy/dt)(=\text{constant } 1)-y(dx/dt)}{x^2}=\frac{x-(y^2/x)}{x^2}=\frac{x^2-y^2}{x^3}$

$=-\frac{24^2}{x^3}=-\frac{24^2}{18^3}=-\frac{8}{81}$ acceleration per second.

III. Results by J. SCHEFFER, A. M., Hagerstown, Md.

If I understand the problem correctly, I find the values to be $\frac{5}{3}$ and $\frac{8}{81}$ instead of $\frac{5}{3}$ and $\frac{1}{12}\frac{3}{1}$.

[NOTE.—A letter from Dr. Byerly states that this, as a number of other errors, crept into the work by oversight. EDITOR.]

MECHANICS.

63. Proposed by A. H. BELL, Hillsboro, Ill.

From a horizontal support at a distance of 10 feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

Comment by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Soon after my solution of the above problem was published Mr. George Richards and Mr. A. H. Bell called my attention to the incorrectness of my assumption in regard to the angles. The error was due to carelessness or inability on my part, either of which is inexcusable. Mr. A. H. Bell sent me the following admirable solution, and I doubt whether a simpler one can be effected. The fact that Mr. Bell has solved this problem is a sure guarantee of its correctness and accuracy.

Let $AB=c=10$, $AD=3=d$, $DC=g=5$, $BC=e=5$, $EB=x+y$, $AE=x-y$.

$$\cos E=\frac{(x+y)^2+(x-y)^2-c^2}{2(x+y)(x-y)}=\frac{(x+y-e)^2+(x-y-d)^2-g^2}{2(x+y-e)(x-y-d)} \quad \ldots\ldots\ldots\ldots (1).$$

$$DF=CN=\frac{c^2+(x-y)^2-(x+y)^2}{2c(x-y)}(x-y-d)=\frac{c^2+(x+y)^2-(x-y)^2}{2c(x+y)}(x+y-e). \quad (2).$$

$$e+d=a=8,\ e-d=b=2.$$

$$\therefore (1)=(4by+g^2-c^2-b^2)x^2-a(4y^2-c^2)x+(c^2+a^2-g^2)y^2-bc^2y$$

$$-\tfrac{1}{4}c^2(a^2-b^2)=0, \text{ or } x^2-\frac{32(y^2-25)}{8y-79}+\frac{139y^2-200y-1500}{8y-79}=0 \ldots (3).$$

$(2) = x^3 - \frac{1}{2}ax^2 - [y^2 - \frac{1}{2}by + (bc^2/8y)]x + \frac{1}{8}ac^2 = 0,$

or $x^3 - 4x^2 - [y^2 - y + (25/y)]x + 100 = 0 \ldots\ldots (4).$

Eliminating x between (3) and (4) we get,

$$128102400y^{12} - 536601600y^{11} - 9985725784y^{10} + 36190002752y^9$$
$$+ 307839235264y^8 - 1004805985048y^7 - 4555231759005y^6$$
$$- 4948451989304y^5 + 108549292200950y^4 + 26216813125200y^3$$
$$- 545379844391 25y^2 + 268623315000y - 49234858937500 = 0 \ldots . (5).$$

The prodigious amount of work necessary to arrive at (5) is wonderful, and great credit is due Mr. Bell for the above solution.

DIOPHANTINE ANALYSIS.

72. Proposed by H. C. WILKES, Skull Run, W. Va.

Given $x^2 + y^2 + z^2 = p^2 + q^2 + r^2$, to find unequal integral values for x, y, z, p, q, and r.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The conditions of the problem are satisfied in the following six identities, in which m, n and r represent any integers, the values being so chosen as to avoid *zero* in the residual quantities,

$$(m+n+r)^2 + (m-r)^2 + (n-r)^2 = (m+n-r)^2 + (m+r)^2 + (n+r)^2,$$
$$(m+n+r)^2 + (m-n)^2 + (n-r)^2 = (m-n+r)^2 + (m+n)^2 + (n+r)^2,$$
$$(m+n+r)^2 + (m-n)^2 + (m-r)^2 = (m-n-r)^2 + (m+n)^2 + (m+r)^2,$$
$$(m+n-r)^2 + (m-n)^2 + (m+r)^2 = (m-n+r)^2 + (m+n)^2 + (m-r)^2,$$
$$(m+n-r)^2 + (m-n)^2 + (n+r)^2 = (m-n-r)^2 + (m+n)^2 + (n-r)^2,$$
$$(m-n+r)^2 + (m-r)^2 + (n+r)^2 = (m-n-r)^2 + (m+r)^2 + (n-r)^2.$$

These equations can be reduced to the following two formulas :

$$\left(\frac{p+2q+2r}{3}\right)^2 + \left(\frac{2p+q-2r}{3}\right)^2 + \left(\frac{2p-2q+r}{3}\right)^2 = p^2 + q^2 + r^2 \ldots\ldots\ldots\ldots (1),$$

$$\left(\frac{2q-p+2r}{3}\right)^2 + \left(\frac{2p+2q-r}{3}\right)^2 + \left(\frac{2p-q+2r}{3}\right)^2 = p^2 + q^2 + r^2 \ldots\ldots\ldots\ldots (2).$$

To insure integral results, assign to p, q, and r multiples of 3.

As 3 things can be arranged in 6 different ways, there may be made, in regard to p, q, and r, 6 different substitutions with each set of assigned values.

In Formula (1), these substitutions will produce 3 different sets of values

when the assigned values are different numbers. In Formula (2), however, only one new set will be obtained for each set of assigned values.

To obtain directly the 3 sets of values by Formula (1), arrange the set of assigned values in the order of their numerical greatness beginning with the largest number; then substitute these respectively for p, q, r; q, p, r; r, q, p.

II. Solution by CHARLES C. CROSS, Libertytown, Md.

$$(m-n+p+q)^2+(m-p-q)^2+(m+n)^2=(m+n-p-q)^2+(m+p+q)^2+(m-n)^2,$$
$$(m+n-p+q)^2+(m-n-q)^2+(m+p)^2=(m-n+p-q)^2+(m+n+q)^2+(m-p)^2,$$
$$(m+n+p-q)^2+(m-n-p)^2+(m+q)^2=(m-n-p+q)^2+(m+n+p)^2+(m-q)^2.$$

Three sets of the sum of three squares equals the sum of three other squares, having no term in common.

Let $m=25$, $n=10$, $p=3$ and $q=1$.

Then $19^2+21^2+35^2=31^2+29^2+15^2$; $33^2+14^2+28^2=17^2+36^2+22^2$, and $37^2+12^2+26^2=13^2+38^2+24^2$.

Three other sets of the sum of three squares equal to the sum of three other squares are given below, but they each have two terms in common.

$$(m+n+p+q)^2+(m-n-p)^2+(m-q)^2=(m-n-p-q)^2+(m+n+p)^2+(m+q)^2,$$
$$(m+n+p+q)^2+(m-p-q)^2+(m-n)^2=(m-n-p-q)^2+(m+p+q)^2+(m+n)^2,$$
$$(m+n+p+q)^2+(m-n-q)^2+(m-p)^2=(m-n-p-q)^2+(m+n+q)^2+(m+p)^2.$$

If we let $m=am$, $n=bn$, $p=cp$, and $q=dq$, the above sets become very general.

III. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

It is manifest that if we find more than one integral value for x, y, and z, in $x^2+y^2+z^2=a$ the question is solved.

Take $x^2+y^2+z^2=p^2$; assume $z^2=2xy$ and reducing we have $x+y=p$, and $y=p-x$, and $z^2=2px-2x^2=\square=$(say) q^2x^2. Then

$$x=\frac{2p}{q^2+2},\quad y=\frac{q^2p}{q^2+2},\quad \text{and } z=\frac{qp}{q^2+2}$$

in which q may be any number. Taking $q=1, 2, 3$, etc., we obtain values of x, y, and z in terms of p, and any number of them that we choose. To obtain integral values, take $p=$the greatest common multiple of the denominators of the values taken, and we have as many values of x, y, and z, the sum of whose squares is constant; and of course the sum of the squares of each set equals the sum of the squares of every other set. For example, take $q=1, 2, 3, 4$.

$$x=\frac{2p}{3},\quad \frac{2p}{6},\quad \frac{2p}{11},\quad \frac{2p}{18}.$$

$$y=\frac{p}{3},\quad \frac{4p}{6},\quad \frac{9p}{11},\quad \frac{16p}{18}.$$

$$z=\frac{2p}{3},\quad \frac{4p}{6},\quad \frac{6p}{11},\quad \frac{8p}{18}.$$

Take $p=198$.

$x=132 \quad 66 \quad 36 \quad 22.$
$y=\ 66 \quad 132 \quad 162 \quad 176.$
$z=132 \quad 132 \quad 108 \quad 88.$

Rejecting the first or second values, we have three solutions of the question. Of course, we may take any two different sets of values and obtain one solution of the question.

IV. Solution by A. H. BELL, Hillsboro, Ill.

Let $x^2+y^2+z^2=p^2+q^2+r^2=(a^2+b^2+e^2)(c^2+d^2)$.

$(c^2+d^2)=\square$.

Then the general value will become $(ac\pm bd)^2+(bc\mp ad)^2+e^2(c^2+d^2)$.

By substitution we obtain 7 sets of values. When paired there are 12 sets of unequal values answering the requirements of this problem.

V. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Let $x=6m$, $y=10m$, $z=11m$, $p=7m$, $q=8m$, $r=12m$.

$\therefore\ x^2+y^2+z^2=p^2+q^2+r^2$.

$\therefore\ (6m)^2+(10m)^2+(11m)^2=(7m)^2+(8m)^2+(12m)^2=257m^2$, where m can have any integral value.

VI. Solution by the PROPOSER.

Let $x=m^2+n^2$, $y=m^2+mn-n^2$, $z=m^2-mn-n^2$.

We have $(m^2+n^2)^2+(m^2+mn-n^2)^2-(m^2-mn-n^2)^2=(m^2+2mn-n^2)^2=p^2$.

$(m^2+n^2)^2+(m^2-mn-n^2)^2-(m^2+mn-n^2)^2=(m^2-2mn-n^2)^2=q^2$.

$(m^2-mn+n^2)^2+(m^2+mn+n^2)-(m^2+n^2)^2=(m^4-4m^2n^2+n^4)=z^2r^2$.

Hence to find integral numbers that will fit the equation, we must find integral values for m and n that will make $(m^4-4m^2n^2+n^4)$ a square.

Let $m^4-4m^2n^2+n^4=r^2$ .. ..(1).

$m^4-4m^2n^2+4n^2=(r+a)^2$ .....(2).

$m^4-2m^2n^2+n^4=(r+b)^2$ .....(3).

From (1)+(2) we have $r=\dfrac{3n^4-a^4}{2a}$.....(4), $m^2=2n^2+\dfrac{3n^4+a^2}{2a}$ ....(5).

From (1)+(3) we have $r=\dfrac{2m^2n^2-b^2}{2b}$...(6), $m^2=n^2+\dfrac{2m^2n^2+b^2}{2b}$ ...(7).

From (4)+(6), $2am^2n^2-3bn^4=ab^2+a^2b$ ....(8).

From (5)+(7), $2am^2n^2-9bn^4-2abn^2=ab^2+a^2b$ ....(9).

Subtracting (9) from (8), $2abn^2=2ab(b-a)$. Whence $b-a=n^2$.

Factoring (8), $n^2(2am^2-3bn^2)=ab(b-a)$. Whence $2am^2-3bn^2=ab$.

Put $2a=n$ and by easy reduction we have $2m^2=b(6n+1)$. Since $\frac{1}{2}(6n+1)$ cannot be a square $b/2$ must be a square to make an integral. Then $m^2=b/2(6n+1)$ is $(n^2+\frac{1}{2}n)/2\times(6n+1)$. $n=4$ being the only value that will make both factors a square, $m=\sqrt{[(n^2+\frac{1}{2}n)/2]\times(6n+1)}=15$.

$\therefore$ $x=241$, $y=269$, $z=149$, $p=329$, $q=89$, $r=191$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

106. Proposed by ELMER SCHUYLER, High Bridge, N. J.

What is the amount of $1000 at compound interest for 3 years at 6%, if it be compounded every instant?

107. Proposed by R. V. ALLEN, Hooker Station, Ohio.

A barn, $ABCD$, length $AB=b$ feet, width $AD=a$ feet, standing in an open field, has a horse tethered to a point, P, in the side, AB, distance $AP=c$ feet, with a rope R feet long. Over what area can the horse graze?

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 10.

ALGEBRA.

94. Proposed by J. W. YOUNG, Columbus, Ohio.

Solve: $\left[\frac{x^2+14x+1}{p^4+14p^2+1}\right]^3=\frac{x(x-1)^4}{p^2(p^2-1)^4}$.

Burnside and Panton's *Theory of Equations*, page 148, ex. 17.

95. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Substitute *numbers* in place of the letters in the following pattern: ..... $\Delta=\sqrt{81^2a^2b^2c^2}=81abc$..... b^2+c^2, a^2+c^2, a^2+b^2; and compute the areas and sides of the whole nest of integral, rational triangles.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

GEOMETRY.

114. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

If a variable ellipse hyperosculate a fixed ellipse at the extremity of the minor axis, the locus of the foci is a circle whose diameter is equal to the radius of curvature.

115. Proposed by MARY BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

The locus of a point such that the sum of the squares of its normals form a given ellipsoid is constant, is a co-axial ellipsoid. [From *C. Smith's Solid Analytical Geometry*, page 95.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 10.

CALCULUS.

85. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A line of double curvature, beginning at some point in the circumference of the base circle of a right cone, winds itself under the constant inclination β to the base circle around the curved surface of the cone. Find its length and that of its projection upon the base circle.

86. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove that the curve whose normal equals its radius of curvature drawn in an opposite direction, is the catenary, $y=c\cosh(x/c)$.

87. Proposed by MARY BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

Integrate $(px-y)(py+x)=h^2p$, where $p=dy/dx$.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

MECHANICS.

82. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

A sphere, diameter $2a$, rests in limiting equilibrium upon the edge of a box and against a vertical wall. If the box be of such dimensions that it will not tip, find the distance of the box from the wall, having given the coefficient of friction between the sphere and wall $\frac{1}{2}$, between the sphere and box $\frac{1}{3}$, and between the box and floor $\frac{3}{4}$. [From Problems in Mechanics proposed to class in Harvard University.]

83. Proposed by MARY BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

A particle is projected upwards in vacuo with a velocity v. Show that on reaching the ground again there is no deviation to the south, but the deviation to the west is $4\omega\cos\lambda(v^3/3g^2)$. [*Laplace*, iv, page 341.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 10.

DIOPHANTINE ANALYSIS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Find three square numbers in harmonical progression.

79. Proposed by EDMUND FISH, Hillsboro, Ill.

Find an integral right triangle in which the bisector of one of the acute angles is also integral.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

MISCELLANEOUS.

72. Proposed by DR. E. D. ROE, JR., Associate Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If a, b, and c are integers, and

$$\left.\begin{matrix} b,\ c-b,\ c-1 \\ c-a-1 \\ c-a-1 \end{matrix}\right\} > 0,$$

$$c-a-b-1 \geq 0,$$

prove that the sum of the series,

$$1+\frac{a.b}{1.c}+\frac{a(a+b).b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}+\cdots\cdots$$

is equal to

$$\frac{(c-1)\,!\,(c-a-b-1)\,!}{(c-a-1)\,!\,(c-b-1)\,!}$$

73. Proposed by CHARLES E. MYERS, Canton, Ohio.

In an ice cream freezer, cream of a homogeneous character and at the uniform temperature of 60° Fahrenheit is put into a cylinder having a closed base, and the whole put into a freezing mixture so as to subject the base and convex surface to a constant temperature of 30° Fahrenheit. Required the temperature at any point within the cream after the expiration of a given time. [From *Higher Mathematics.*]

74. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

The longest diameter of a horizontal ellipse is $CB=2a=6$ feet. Its shortest diameter is $EF=2b=4$ feet, their intersection being at D. Find in an indefinite vertical plane passing through CB, a point A 5 feet$=c$ from D, the ellipse being seen from A as a circle.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

EDITORIALS.

Dr. Robert J. Aley, of the University of Indiana, has been elected to membership in the *Deutsche Mathematiker-Vereinigung*, and also in the *London Mathematical Society*.

This issue has been somewhat delayed by the illness of the editor. We shall make strenuous efforts to have all subsequent numbers reach our subscribers by the last of each month.

We are following our previous plan of sending out the January number to each of our old subscribers. Any one wishing to discontinue should return this number with his name and address legibly written on the wrapper.

Contributors should observe the following in sending in contributions: 1. Write only on one side of the paper; 2. Sign your name and address to each contribution; 3. In contributing problems or solutions, sign your name to

each problem or solution, and let each problem or solution be on a separate sheet of paper. By observing these directions, your contributions will often be saved from the waste basket.

BOOKS AND PERIODICALS.

A Text-Book of Statics. By William Briggs, M. A., LL. B., F. R. A. S., General Editor of the Tutorial Series, Principal of University Correspondence College ; and G. H. Bryan, M. A., Smith's Prizeman, Fellow St. Peter's College, Cambridge. 8vo. Cloth, 220 pages. Price, 3*s.* 6*d.* New York : Hinds & Noble, Publishers.

This little work presents in a very excellent manner all the facts and principles in Statics that can be mastered by a student in under graduate work. All principles are clearly presented and illustrated by excellent diagrams. Many illustrative problems are solved and many problems are inserted at the end of each chapter. B. F. F.

Elements of Trigonometry, Plane and Spherical. By Andrew W. Phillips, Ph. D., and Wendell M. Strong, Ph. D., Yale University. 8vo. Cloth, 138 pages. Price, 90 cents. New York : Harper & Brothers.

This work, in addition to the usual matter treated, contains several features which are entirely distinctive: First, in Spherical Trigonometry, we have the photographic reproduction of models used in Yale University. These are beautiful in themselves and add a charm to the book only equaled by that of the Elements of Geometry by Professors Phillips and Fisher. Second, the graphic representation of the Trigonometric and Inverse Trigonometric Functions. Third, the brief treatment of Plane, Spherical and Pseudo-Spherical Trigonometries. In this treatment it is pointed out that Plane Trigonometry is a special case of Spherical Trigonometry, or better, is the limiting case of Spherical Trigonometry and Pseudo-Spherical Trigonometry. This discussion might have been enlarged upon to advantage, but not, however, without the use of some principles of the Calculus.

The printed page does not have the artistic appearance which the work deserves, it presenting the appearance of being printed from old type. B. F. F.

On the Study and Difficulties of Mathematics. By Augustus De Morgan. New Edition. 8vo. Cloth, 288 pages. Price, $1.00. Chicago : The Open Court Publishing Co.

This work is of special interest to all teachers of students of mathematics. In it are explained all the difficulties that arise in the study of elementary mathematics, making it possible for the student to master the first principles of mathematics in a way that will make the study of higher mathematics a joy forever, in that he will not need to continually return to unlearn principles taught him in the elements. The chapter on Arithmetical Fractions should be read several times, as it makes very clear what has been a necessity in the development and progress of mathematics, viz., the carrying of terms in that which is simple to that which is complex, and enlarging their meaning as our ideas enlarge. A failure to grasp this important principle has led to endless and useless discussions; as, for example, whether 3 and $3\sqrt{-1}$ are numbers.

The whole work bears the impress of its author's genius. It is a notable instance of a mathematician of eminent mathematical attainment setting himself the task of ridding

the elementary branches of mathematics of some of their inaccuracies and solecisms. Great credit is due the editor, Mr. T. J. McCormack. for his service to the cause of mathematics in bringing out new editions of such valuable works. B. F. F.

The Mathematical Magazine. A Journal of Elementary and Higher Mathematics. Edited and published by Artemas Martin, M. A., Ph. D., LL. D., Washington, D. C. Price, $1.00 for four numbers. Single number, 30 cents.

The number for December, 1898, contains the following: About Fifth Power Numbers Whose Sum is a Fifth Power, by Dr. Martin; Notes About Square Numbers Whose Sum is Either a Square or the Sum of Other Squares, by Dr. Martin; Formulas for the Sides of Rational Plane Triangles, by Dr. Martin; A Method of Finding, without Tables, the Number Corresponding to a Given Logarithm, by Dr. Martin. These papers occupy 40 pages; 16 pages are devoted to the solution of problems and 9 pages to Problems, Editorial Notes, and Book Reviews. This is the largest single number of the *Magazine* that has yet appeared. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

The February number of *The Cosmopolitan* maintains its high standard of artistic excellence and literary merit. B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., New York.

"The Progress of the World," the editorial department of the February *Review of Reviews*, deals this month with the new problems of colonial administration now confronting the country, with the Senatorial campaigns in the different States, with the polygamy question, with the question of army beef in its bearings on the reorganization of the War Department, with our recent industrial progress, protective tariffs, and the "trusts,"and with the month's developments in foreign politics. B. F. F.

Errata.

In Professor Hoover's solution of problem 70, Mechanics, page 275, where he says $k^2=\frac{1}{2}r^2$, he should have said $k^2=\frac{3}{5}r^2$. $k^2=\frac{1}{2}r^2$ is for the cylinder. This error was pointed out by Professor Scheffer.

Prof. R. E. Gaines called attention to an error in Professor Zerr's solution of problem 71, Mechanics, page 275. A correct solution of this problem will appear in a later issue of the Monthly.

The following errors were pointed out by Prof. W. F. Bradbury.

Page 292, line 6, "$y=2$" should be $x=2$.

Page 293, line 13, "$2-\frac{1}{2(x+3)}$" should be $2-\frac{1}{2(y+3)}$.

Page 296, line 15, "$\therefore A=$" should be $AB=$.

Page 296, line 23, "$2\theta x=\frac{1}{2}\pi$," etc., should be $2\theta x=\pi$, etc.....(3).

Page 296, line 25, "$-x^6/4^2.5$" should be $+x^6/4.5$.....(5).

Page 298, line 13, "acute acute" should be acute angle.

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ON SYMMETRIC FUNCTIONS.

By DR. E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from January Number.]

This term is ever found in

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} p_{i_1} p_{i_2} p_{i_3}$$

where p_{i_1} is to be applied to the j_1st line, p_{i_2} to the j_2d line, and p_{i_3} to the j_3d line. The term then corresponds to the substitution $\begin{pmatrix} j_1 & j_2 & j_3 \\ 1 & 2 & 3 \end{pmatrix}$, and has the same sign as before. We are now ready to state the general case.

g. We must show that it is indifferent whether we apply the exponents in all possible permutations to the columns or to the rows of D, *i. e.*, that to every term of the second formation corresponds the same term in the first formation, and then since the second formation is the straightforward product of D and Σ, it will follow that the first formation is equal to the product of D and Σ. We may denote the second and first formations by

$$\begin{vmatrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \ldots\ldots \alpha_1^{n-1} \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \ldots\ldots \alpha_2^{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n^0 & \alpha_n & \alpha_n^2 \ldots\ldots \alpha_n^{n-1} \end{vmatrix} \begin{matrix} p_{i_1} \\ p_{i_2} \\ \vdots \\ p_{i_n} \end{matrix} \text{ and } \begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \ldots\ldots \alpha_n^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \ldots\ldots \alpha_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} \ldots\ldots \alpha_n \end{vmatrix} p_{i_1} p_{i_2} \ldots p_{i_n}.$$

$i_1, i_2, \ldots\ldots i_n$ form a permutation of the numbers 1, 2, 3, ... n, and the expression on the left signifies that D has been multiplied by $\alpha_1^{p_{i_1}}\alpha_2^{p_{i_2}}\alpha_n^{p_{i_n}}$ of $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$, the first row by $\alpha_1^{p_{i_1}}$, the second by $\alpha_2^{p_{i_2}}$, ... the nth by $\alpha_n^{p_{i_n}}$, or more briefly that p_{i_1} is to be applied as exponent to the elements of the first row, p_{i_2} in the same way to those of the second, ... and p_{i_n} similarly to those of the nth row, while the expression on the right (D with columns changed into rows and horizontal line of p's) shall signify that the p's are to be applied arbitrarily as exponents to the lines of D as written, the same thing as applying them to the columns as before written, and in such order that p_{i_1} is applied to the j_1st line, p_{i_2} to the j_2d line, $\ldots\ldots p_{i_n}$ to the j_nth line.

h. Any term $\alpha_1^{p_{i_1}+j_1-1}\alpha_2^{p_{i_2}+j_2-1}\alpha_3^{p_{i_3}+j_3-1}\ldots\ldots\alpha_n^{p_{i_n}+j_n-1}$ (an expression which is seen to be an $(n\,!)^2$ valued function when one permutes the n p's in all possible ways, and also the n j's) correspond, with reference to the expression on the left hand, to the $(n\,!)^2$ valued substitution

$$s_1 s_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ j_1 & j_2 & j_3 & \ldots\ldots & j_n \end{pmatrix}$$

the first factor referring to the vertical series of p's, the second afterwards to the determinant, when the indicated multiplications have been performed. With reference to the right hand expression, corresponds the $(n\,!)^2$ valued substitution

$$s_1^1 s_2^1 = \begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ 1 & 2 & 3 & \ldots\ldots & n \end{pmatrix},$$

in a similar manner, the first factor referring to the horizontal series of p's, and the second to the determinant after the p's have been applied to the elements, to the same term. It is clear that, numerically at least, the term is the same, for we must apply p_{i_1} to the j_1st line, and then take the first column ; this gives $\alpha_1^{p_{i_1}+j_1-1}$; next we must apply p_{i_2} to the j_2d line, and take the second column ; this gives $\alpha_2^{p_{i_2}+j_2-1}$; in general we must apply p_{i_r} to the j_rth line, and then take the rth column ; this gives $\alpha_r^{p_{i_r}+j_r-1}$, and the product $\prod_{r=1}^{r=n}\alpha_r^{p_{i_r}+j_r-1}$ is numerically the term in question. The first substitutions s_1 and s_1^1 in either case have no effect on the sign of the term ; they merely assign the p's to their proper lines. The second substitutions

$$s_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ j_1 & j_2 & j_3 & \ldots\ldots & j_n \end{pmatrix} \text{ and } s_2^1 = \begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ 1 & 2 & 3 & \ldots\ldots & n \end{pmatrix}$$

are reciprocals and have the same modulus or sign factor.

i. We have proved : The compound substitution $s_1 s_2$ corresponds to all the $(n\,!)^2$ terms which $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$ is capable of having. Similarly the compound substitution $s_1^1 s_2^1$ which is simultaneous with $s_1 s_2$ and depends thereon corresponds to the $(n\,!)^2$ terms of $(p_1 p_2 \ldots\ldots p_n)$ and the terms corres-